

Counting maps from curves to projective space via graph theory

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1 Brill–Noether theory on reducible curves

In Brill–Noether theory, one studies linear series on curves, in order to understand when a curve C of genus g comes equipped with a nondegenerate morphism of degree d to \mathbb{P}^r . For a *general* curve $[C] \in \mathcal{M}_g$, a basic answer is provided by the *Brill–Noether theorem* of Griffiths and Harris, which establishes that C admits such a morphism if and only if the invariant

$$\rho(d, g, r) = g - (r + 1)(g - d + r)$$

is nonnegative, in which case ρ also computes the dimension of the space of linear series g_d^r of degree d and rank r on C .

The Brill–Noether question also admits natural extensions, obtained by imposing incidence conditions on the images of the linear series in question. Namely, given integers $m \geq d$ and $s \geq d - r$, let $\mu(d, r, s) := d - r(s + 1 - d + r)$ denote the virtual dimension of space of inclusions

$$g_{m-d}^{s-d+r} + p_1 + \cdots + p_d \hookrightarrow g_m^s \quad (1.1)$$

on a fixed curve. When the curve C in question is smooth, and the g_m^s is a subspace $V \subset H^0(C, L)$ of global sections of a line bundle L , such inclusions correspond to d -tuples of points $p_1, \dots, p_d \in C$ for which the natural evaluation map

$$\text{ev} : V \rightarrow H^0(C, L/L(-p_1 - \cdots - p_d)) \quad (1.2)$$

satisfies $\text{rank}(\text{ev}) = d - r$. Geometrically, such d -tuples determine d -secant $(d - r - 1)$ -planes to the image of the g_m^s . In [3], we showed that when $\rho = 0$ and $\mu < 0$, there are *no* inclusions (1.1) on a general curve:

Theorem 1.1. *If $\rho = 0$ and $\mu < 0$, then a general curve C admits no linear series g_m^s with d -secant $(d - r - 1)$ -planes.*

Our proof of Theorem 1.1 is a natural generalization of the Brill–Noether proof given in [5, Ch. 5] and is based on an analysis of (limit) linear series on certain reducible curves of compact type.

2 Counting secant planes via graph theory

An immediate corollary of Theorem 1.1 is that *when $\rho = 0$ and $\mu = -1$, curves with linear series g_m^s with d -secant $(d - r - 1)$ -planes determine a divisor in \mathcal{M}_g* . The case $r = 1$ is particularly natural: in that case, exceptional secant planes correspond to d -tuples of points for which the evaluation maps (1.2) fail to be *surjective*. We show [4, Thm 2]:

Theorem 2.1. *The coefficients of the homology classes of secant-plane divisors in $\overline{\mathcal{M}}_g$, realized as linear combinations of standard generators over \mathbb{Q} , are explicit linear combinations of hypergeometric series of type ${}_3F_2$.*

The key ingredient for proving Theorem 2, which is of interest in its own right, is the following auxiliary result [3, Thm 4]:

Theorem 2.2. *The generating series for the virtual number N_d of d -secant $(d - 2)$ -planes to a degree- m curve C of genus g in \mathbb{P}^{2d-2} is*

$$\sum_{d \geq 0} N_d(g, m) = \left(\frac{2}{(1 + 4z)^{1/2} + 1} \right)^{2g-2-m} \cdot (1 + 4z)^{\frac{g-1}{2}}. \quad (2.1)$$

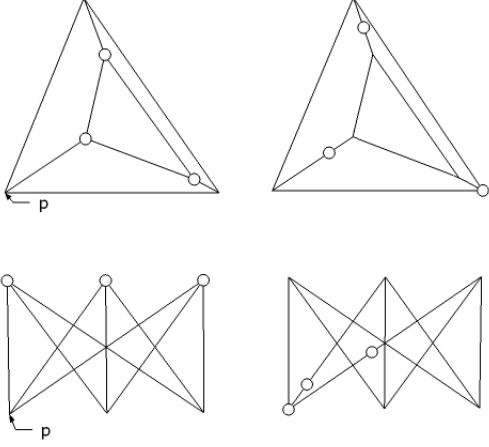
Two ingredients enter into our proof of Theorem 2.2. The first is *Porteous' formula*, which computes the homology class of the locus of d -secant $(d - 2)$ -planes as a determinant in the Chern classes of the so-called d th *tautological* bundle $L^{[d]}$ over the d th Cartesian product C^d , whose fiber over $(p_1, \dots, p_d) \in C^d$ is $H^0(L/L(-p_1 - \dots - p_d))$. The second is a combinatorial analysis of the resulting intersection-theoretic formula, which amounts to a weighted count of subgraphs of the complete graph K_d on d vertices.

3 Linear series on metric graphs

In the preceding section, graphs naturally arose in connection with counting (secant planes to) morphisms via the formalism of intersection theory. But graph theory also intervenes in a natural way as a result of degeneration, via the passage from a nodal curve to the *dual graph* recording the incidences of its components.

There is a theory of complete linear series on metric graphs with \mathbb{R} -valued edge lengths due to Baker–Norine [1] and Mikhalkin–Zharkov [6]. Concretely, a (complete) linear series $|D|$ on a metric graph Γ is a configuration D of points in Γ , modulo an equivalence relation defined by piecewise-linear functions. Moreover, there is an explicit combinatorial *burning algorithm* due to Dhar for computing the rank of a configuration of points $D \in \text{Div}(\Gamma)$; see [2].

Contrasting examples in genus four. Figure (a) shows two metric graphs of genus 4 (here, as in the remainder of the article, we assume that all weights on vertices are 0). The



(a) Evolutions of degree-3 configurations along the planar “wheel” graph and $K_{3,3}$

top graph Γ_1 pictured is planar, and the 3 circles determine a degree-3 configuration D_1 of trivial rank. Indeed, a fire that burns from p will be repelled by the 3 points in the support of D_1 , which then evolve at equal velocity against the incoming fire. Assuming the planar graph has generic edge lengths, a single point p_1 of D_1 will arrive at a vertex v_1 of Γ_1 , at which point a fire burning from p will approach v_1 (and p_1) from 2 distinct directions and all of Γ_1 will burn. By contrast, the configuration D_2 of 3 points on the complete bipartite graph $\Gamma_2 = K_{3,3}$ evolves in such a way that at no time will any fire based at any point p approach any point in the support of D_2 along two distinct directions. It follows that $r(D_1) \geq 1$, and in fact the rank of D_1 is precisely 1.

4 The gonality of tree-decomposed graphs

The contrast between the behavior of degree-3 configurations on the planar genus-4 graph Γ_1 and on $\Gamma_2 = K_{3,3}$ is instructive. In fact, it is not hard to check that Γ_1 and Γ_2 each admit two degree-3 configurations of rank 1, as predicted by Brill–Noether theory for curves of genus 4. However, on Γ_1 , these configurations depend strongly on the metric structure: each is obtained by placing 2 points on 2 out of 3 inner (resp., outer) “rim” vertices, and a third point along a “spoke” at distance from an outer (resp., inner) vertex at distance equal to the length of the shortest spoke. On Γ_2 , on the other hand, each rank-1 configuration is associated to a choice of one of the two sets of 3 vertices along which Γ_2 decomposes as a union of three 4-edged trees.

Definition/construction. Let $V = \{v_1, \dots, v_n\}$, $n \geq 3$ denote a fixed set of vertices, and let T_1, T_2 , and T_3 denote three trees each containing V as vertices but which are otherwise pairwise disjoint. The three trees T_i , $1 \leq i \leq 3$ glue naturally to a graph Γ ; we say that Γ admits a *tree decomposition* (T_1, T_2, T_3) *rooted along* V .

Some of the most famous graphs of genus at most 10 admit such tree decompositions: besides $K_{3,3}$, the examples of the so-called Petersen, Heawood, and Pappus graphs in genera 6, 8, and 10 (respectively) are tree-decomposable.

Theorem 4.1 (Existence of rank-one series on tree-decomposed graphs). *Suppose that the metric graph Γ admits a tree decomposition rooted on $n \geq 3$ vertices V . Then V determines a rank-one, degree- n divisor D on Γ .*

Proof. The result follows from the burning algorithm. Namely, fix any choice of base point p from which to burn, say $p \in T_1$ without loss of generality. Any fire burning from p along T_1 is repelled by the points p_1, \dots, p_n of D supported along V , which then evolve at equal velocity along T_1 away from V . The burning process iterates until ultimately the fire is extinguished by at least one of the points p_i , which proves that $r(D) \geq 1$. Similarly, to prove that $r(D) < 2$, it suffices to allow *two* successive fires to burn from p_1 : the first fire simply has the effect of canceling out p_1 , while the second burns through all of Γ . \square

Definition. A graph (or a curve) Γ of genus g is n -gonal whenever $n = \min\{j \in \mathbb{Z}_{>0} : \exists \text{ a } g_j^1 \text{ on } \Gamma\}$.

Theorem 4.2. $K_{3,3}$, Petersen, Heawood, and Pappus are 3-gonal, 4-gonal, 5-gonal, and 6-gonal graphs, respectively.

Proof sketch. It is easy to exhibit tree decompositions of these graphs rooted on $n = 3, 4, 5$, and 6 vertices, respectively. Whence, by Theorem 4.1, it suffices to prove that each of these n -rooted tree-decomposed graphs admits no degree- $(n - 1)$ configurations D of positive rank. Replacing D by a linearly equivalent configuration if necessary, we may assume that each point in $\text{Supp}(D)$ appears with multiplicity at most 2. It remains to carry out a case-by-case inspection using the burning algorithm. \square

It is not hard to produce graphs that decompose as unions of trees rooted on $n \geq 3$ vertices but are α -gonal with $\alpha < n$. So additional conditions are needed to ensure that n -gonality is achieved. Theorem 4.2 and experimentation give some evidence that it suffices to maximize the minimal cycle length, or *girth*, of Γ .

Conjecture 4.1. *A metric graph Γ that admits a tree-decomposition (T_1, T_2, T_3) rooted on n vertices is n -gonal provided $\text{girth}(\Gamma)$ is maximal for the combinatorial type of (T_1, T_2, T_3) .*

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References

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